Updating Circuit Theory - Composite Conductors

Ian Darney

A method of simulating the distribution of current on the surfaces of conductors of a cable-form is described. This is dramatically more useful to designers than multi-coloured images of electromagnetic fields.

This is done by representing the surface of each conductor as an array of elemental conductors. The technique is introduced here by simulating the current distribution in two parallel conductors when these are short-circuited at one end and a voltage step of 1 Volt is applied at the other. Calculating the sum of the currents in the elemental conductors of each composite gives the total current in that conductor. There is then enough data to calculate the characteristic impedance, inductance, and capacitance of each conductor, and a circuit model of this simple assembly is derived.

The value of the capacitance between the conductors is shown to be precisely the same as that derived in books on Electromagnetic Theory. This provides confidence in the technique. The method can be developed to create a circuit model of any cable assembly.

The computations involve the use of a Mathcad worksheet; and sections of that worksheet are displayed at appropriate places in this article. The programs are easy to understand, since they are written in the same way as a set of equations in a textbook on mathematics. The illustrations of current distribution were created by transferring data from the worksheet to Microsoft Excel and invoking the 'Bubble Plot' option

The formulae used in the equations are derived in the article 'The three conductor model', which is accessible for download from the Electronics World website.

The technique described in this article is a development of the method used by Culham Lightning Studies Unit to analyse the effect of lightning strikes on aircraft electronic systems.

Concept

The interaction between the external electromagnetic field and charges on any conductor takes place at the surface of that conductor. Since the charge intensity on that surface is proportional to the strength of the electromagnetic field at that location, one way of analysing the effect of that field is to simulate the way charges are distributed across the surface. This can be done by replacing each conductor with an array of elemental conductors, as illustrated by Figure 1.

Each elemental conductor represents a small segment, *s*, of the surface. Hence

$$s = 2 \cdot \pi \cdot r = \theta \cdot Rad \qquad (1)$$

where *r* is the radius of the elemental conductor and *Rad* is the radius of the composite conductor.

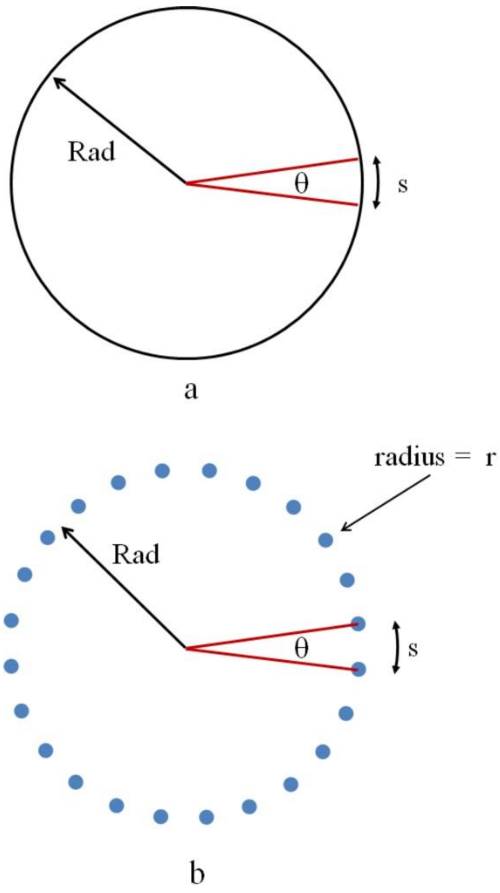


Figure 1 Concept of the composite conductor
a Cross-section of the actual conductor
b Array of elemental conductors

In this example, it is assumed that the conductor is isolated. So, the charge is distributed evenly round the surface. If *n* is the number of elemental conductors:

$$n \cdot \theta = 2 \cdot \pi$$

and

$$r = \frac{Rad}{n} \tag{2}$$

Twin conductor assembly

An array of 24 elemental conductors is illustrated by Figure 2. This represents the cross-section of two conductors spaced 4 mm apart. The radius of each conductor is 1mm. This assembly can be defined using a table of three columns; recording the x co-ordinates, the y co-ordinates, and the radius of each elemental conductor.

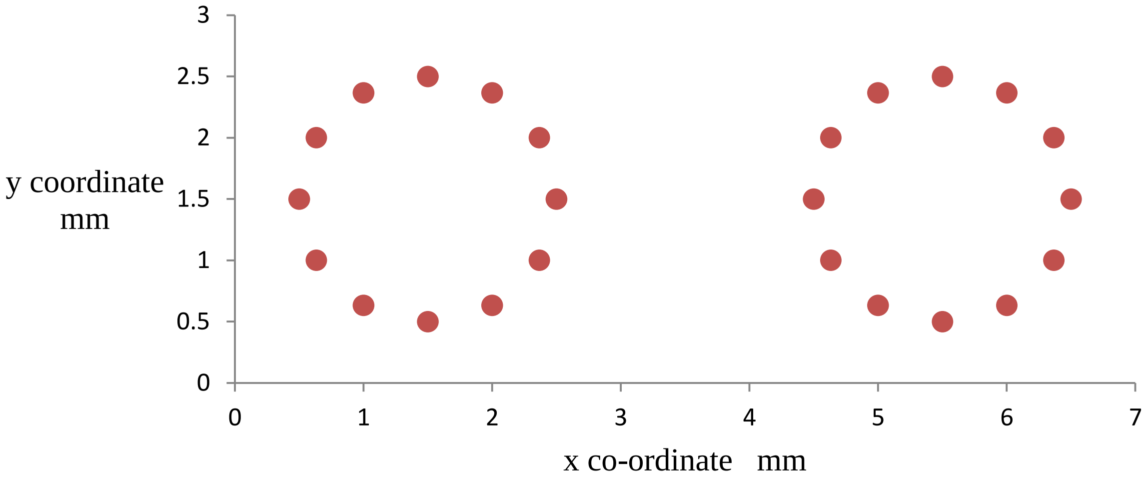


Figure 2 Using elemental conductors to simulate a twin-conductor assembly

In the article 'The three conductor model' a general formula is derived for primitive impedance:

$$Zp_{i,j} = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{\mu}{\varepsilon}} \cdot \ln\left(\frac{l}{r_{i,j}}\right) \quad (3)$$

where $r_{i,j}$ is the separation between conductors i and j, $r_{i,i}$ is the radius of conductor i, μ is the permeability of the conductor, ε is the average permittivity of the insulation, and l is the length of the assembly.

It is assumed that a voltage of 1 Volt is applied along the length. Figure 3 is an extract from a Mathcad worksheet. This calculates the primitive impedances associated with each elemental conductor and records them in an array ***Zp***. It then defines the voltage vector ***Vp*** and calculates the value of all the currents in the elemental conductors. The vector ***Ip*** records the value of the current in every elemental conductor.

$\mu := 4 \cdot \pi \cdot 10^{-7}$ $\quad \varepsilon := 8.854 \cdot 10^{-12}$ Fundamental constants

$n1 = 12 \quad n2 = 12$ Number of elemental conductors in each composite

$N = 24$ Total number of elemental conductors

$$\boldsymbol{Zp} := \left| \begin{array}{l} H \leftarrow \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{\mu}{\varepsilon}} \\ \text{for } i \in 1..N \\ \quad \text{for } j \in 1..N \\ \quad \quad \left| \begin{array}{l} h \leftarrow \boldsymbol{x}_j - \boldsymbol{x}_i \\ v \leftarrow \boldsymbol{y}_j - \boldsymbol{y}_i \\ rad \leftarrow \sqrt{h^2 + v^2} \\ rad \leftarrow \boldsymbol{r}_i \text{ if } rad = 0 \\ \boldsymbol{Z}_{i,j} \leftarrow H \cdot ln\left(\frac{1}{rad}\right) \end{array} \right. \\ \boldsymbol{Z} \end{array} \right.$$

Calculating the primitive impedance of every elemental conductor.

The first action is to calculate the separation between every conductor and every other conductor, *rad*.
Then this value is used to calculate the value of the primitive impedance
The output of this subroutine is the vector ***Z***
Its contents are transferred to ***Zp***

$i := 1..N$

$\boldsymbol{Vp}_i := 1$ Setting a voltage of 1V along every elemental conductor

$\boldsymbol{Ip} := lsolve(\boldsymbol{Zp}, \boldsymbol{Vp})$ Calculating the value of the current along every element

Figure 3 Calculating values for the currents in the elemental conductors

The distribution of current is illustrated by the bubble plot of Figure 4. Since the same voltage is applied across each conductor, all the currents are flowing in the same direction. That is, current flow is concentrated on the outward facing surfaces.

This can be visualised in two ways:

- the current due to the magnetic field in the environment.
  or
- the charge distribution due to the electric field in the environment.

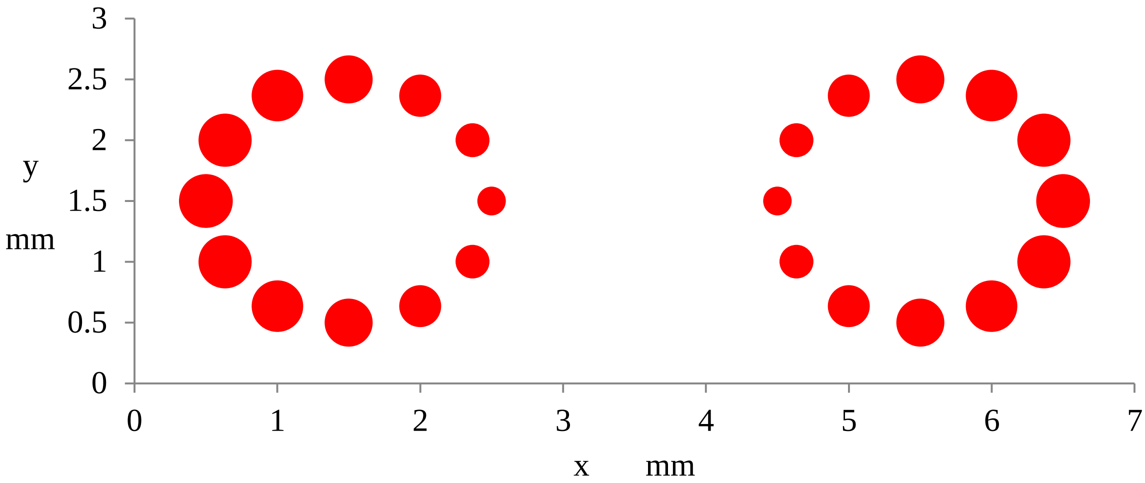


Figure 4 Distribution of antenna current in two parallel conductors

Differential mode current

To analyse differential mode current, it is assumed that the terminals at the far end of the cable are shorted together and that a voltage is applied between the two conductors at the near end. For the array of Figure 2, this voltage appears between elements 12 and 13 at the near end. Figure 5 illustrates this.

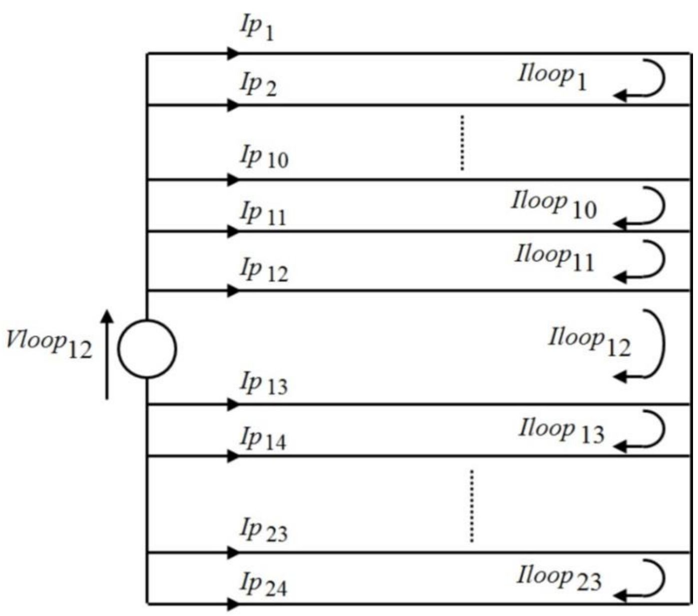


Figure 5 Definition of loop currents and voltages

To calculate the values of the current *Ip* in each elemental conductor, it is first necessary to calculate the loop currents illustrated on Figure 5. The section of the Mathcad worksheet which performs this function is illustrated by Figure 6.

$$\boldsymbol{Zloop} := \left| \begin{array}{l} \text{for } i \in 1..N-1 \\ \quad \text{for } j \in 1..N-1 \\ \quad\quad \boldsymbol{Z}_{i,j} \leftarrow \boldsymbol{Zp}_{i,j} - \boldsymbol{Zp}_{i,j+1} - \boldsymbol{Zp}_{i+1,j} + \boldsymbol{Zp}_{i+1,j+1} \\ \boldsymbol{Z} \end{array} \right.$$

Calculating the loop impedance of every pair of elemental conductors

$$\boldsymbol{Vloop} := \left| \begin{array}{l} \text{for } i \in 1..N-1 \\ \quad \left| \begin{array}{l} \boldsymbol{V}_i \leftarrow 0 \\ \boldsymbol{V}_i \leftarrow 1 \ \text{ if } i = n1 \end{array} \right. \\ \boldsymbol{V} \end{array} \right.$$

locating the voltage source in loop *n1*

$$\boldsymbol{Iloop} := lsolve(\boldsymbol{Zloop}, \boldsymbol{Vloop})$$

Calculating the value of the current in every loop.

$$\boldsymbol{Ip} := \left| \begin{array}{l} \boldsymbol{I}_1 \leftarrow \boldsymbol{Iloop}_1 \\ \text{for } i \in 2..N-1 \\ \quad \boldsymbol{I}_i \leftarrow \boldsymbol{Iloop}_i - \boldsymbol{Iloop}_{i-1} \\ \boldsymbol{I}_N \leftarrow -\boldsymbol{Iloop}_{N-1} \\ \boldsymbol{I} \end{array} \right.$$

Calculating the value of the current flowing along every elemental conductor.

Figure 6 Calculating the values of the currents ***Ip*** in the circuit of Figure 5.

Each loop impedance is due to current flowing in each conductor of the loop, plus the mutual coupling between the two conductors of that loop. This results in the vector ***Zloop***. The voltage at the near end between conductors 12 and 13 is defined as one volt. All the other voltages at the near end are zero. This is defined by the vector ***Vloop***. The third function of Figure 6 calculates the values of the loop currents and stores them in the vector ***Iloop***. The final action of this part of the worksheet is to calculate the value of the current in each elemental conductor and store it in the vector ***Ip***.

Figure 7 illustrates the current distribution in the assembly. The area of each red circle is proportional to the current flowing in the elemental conductor at that location. This current is flowing from left to right, as illustrated by the currents *Ip* in Figure 5. Each blue circle represents the magnitude of the current flowing from right to left in that diagram.

This depiction can be visualised as the current distribution in the send and return conductors when constant current is flowing in the loop, or as the distribution of charges when these conductors are isolated from each other.

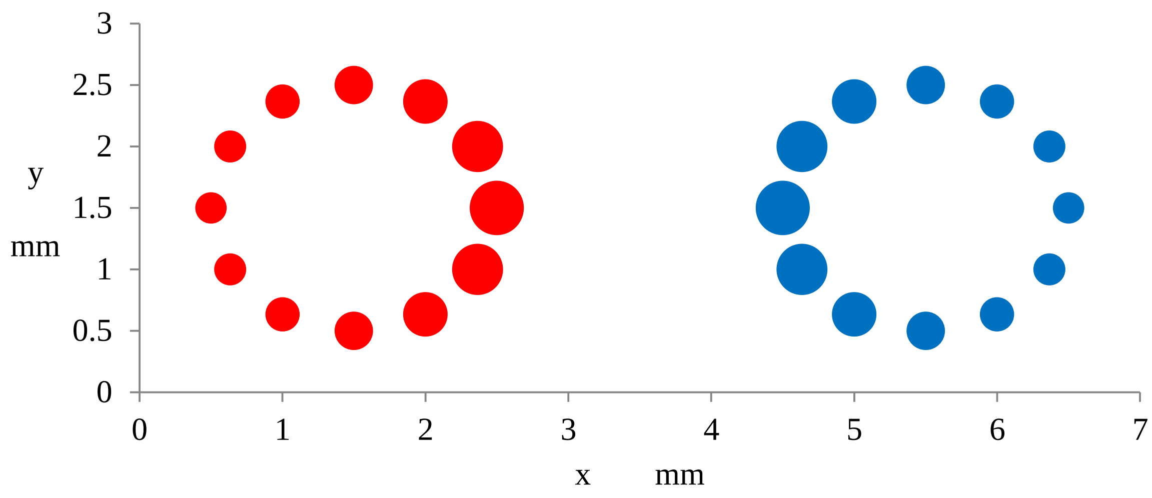


Figure 7 Distribution of differential-mode current in the two composite conductors

Component values.

The worksheet depicted by Figure 8 first defines the elemental conductors which simulate composite conductors 1 and 2, then calculates the total current in each composite and stores them in the vector ***Ic***. This computation indicates that 6.33 mA flows along the send conductor and back along the return conductor when a voltage of 1 Volt is applied to the input terminals.

The next step in the computation is to calculate the sum of the voltages along every elemental conductor of the send conductor. Since the voltage along each elemental conductor is 0.5 volts and there are 12 such conductors, it is necessary to divide the total voltage by 12. Similarly, the voltage along the return conductor is calculated to be -0.5 Volt. This data is recorded in the vector ***Vc***.

It is then a simple task to calculate the value of the impedances $\boldsymbol{Zc}_1$ and $\boldsymbol{Zc}_2$ presented by each composite conductor to the input voltage.

For a lossless line, the characteristic impedance is

$$Zc = \sqrt{\frac{Lc}{Cc}} \tag{4}$$

and the time taken for a step current to propagate from one end of the line to the other is:

$$T = \sqrt{Lc \cdot Cc} \tag{5}$$

Both these relationships are derived in the article ‘Transmission Line Model’.

$$\textbf{Start} := \begin{pmatrix} 1 \\ \boldsymbol{n}_1 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \end{pmatrix}$$

defining the first elemental conductor in each composite conductor

$$\textbf{End} := \begin{pmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_1 + \boldsymbol{n}_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \end{pmatrix}$$

defining the last elemental conductor in each composite conductor

$$\begin{aligned} \textbf{Ic} := \; & \text{for } h \in 1..2 \\ & \quad S \leftarrow 0 \\ & \quad \text{for } i \in \textbf{Start}_h .. \textbf{End}_h \\ & \quad \quad S \leftarrow S + \textbf{Ip}_i \\ & \quad \boldsymbol{I}_h \leftarrow S \\ & \boldsymbol{I} \end{aligned}$$

Calculating the total current along each composite conductor

$$\textbf{Ic} = \begin{pmatrix} 6.331 \times 10^{-3} \\ -6.331 \times 10^{-3} \end{pmatrix}$$

$$\begin{aligned} \textbf{Vc} := \; & \text{for } h \in 1..2 \\ & \quad S \leftarrow 0 \\ & \quad \text{for } i \in \textbf{Start}_h .. \textbf{End}_h \\ & \quad \quad \text{for } j \in 1..N \\ & \quad \quad \quad S \leftarrow S + \textbf{Ip}_j \cdot \textbf{Zp}_{i,j} \\ & \quad \boldsymbol{V}_h \leftarrow S \\ & \frac{\boldsymbol{V}}{\boldsymbol{n}_h} \end{aligned}$$

Selecting all the primitive impedances associated with composite conductor h multiplying each by every primitive current, adding up all these voltage increments and putting the total in the element $\boldsymbol{V}_h$
Since there are $\boldsymbol{n}_h$ elemental conductors in composite h, the sum total is divided by $\boldsymbol{n}_h$
This gives the voltage drop along each composite conductor

$$\textbf{Vc} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$$

$$\textbf{Zc} := \frac{\textbf{Vc}}{\textbf{Ic}} = \begin{pmatrix} 78.972 \\ 78.972 \end{pmatrix}$$

Characteristic impedance of each composite conductor

Figure 8 Computing the values for the circuit impedances

From equation (4),

$$Cc = \frac{Lc}{Zc^2} \tag{6}$$

Substituting for Cc in (5)

$$T = \frac{Lc}{Zc} \tag{7}$$

giving

$$Lc = T \cdot Zc \tag{8}$$

substituting for *Lc* in (6)

$$Cc = \frac{T}{Zc} \tag{9}$$

The section of the worksheet which assigns values to these parameters is illustrated by Figure 9.

$T := \sqrt{\mu \cdot \varepsilon} = 3.336 \times 10^{-9}$ Time taken for a step voltage to propagate along a length of 1 metre of the cable

$\boldsymbol{Lc} := T \cdot \boldsymbol{Zc} = \begin{pmatrix} 2.634 \times 10^{-7} \\ 2.634 \times 10^{-7} \end{pmatrix}$ Inductance of each composite conductor

$\boldsymbol{Cc} := \dfrac{T}{\boldsymbol{Zc}} = \begin{pmatrix} 4.224 \times 10^{-11} \\ 4.224 \times 10^{-11} \end{pmatrix}$ Capacitance of each composite conductor

From the book Electromagnetic Concepts and applications by G.G. Skitek and S. V. Marshall, pages 208 to 209: Capacitance between two cylindrical conductors.

$b := 2 \cdot 10^{-3}$ $r := 1 \cdot 10^{-3}$ $l := 1$

$$Ctheory := \frac{\pi \cdot \varepsilon \cdot l}{ln\left(\dfrac{b + \sqrt{b^2 - r^2}}{r}\right)} = 2.112 \times 10^{-11}$$

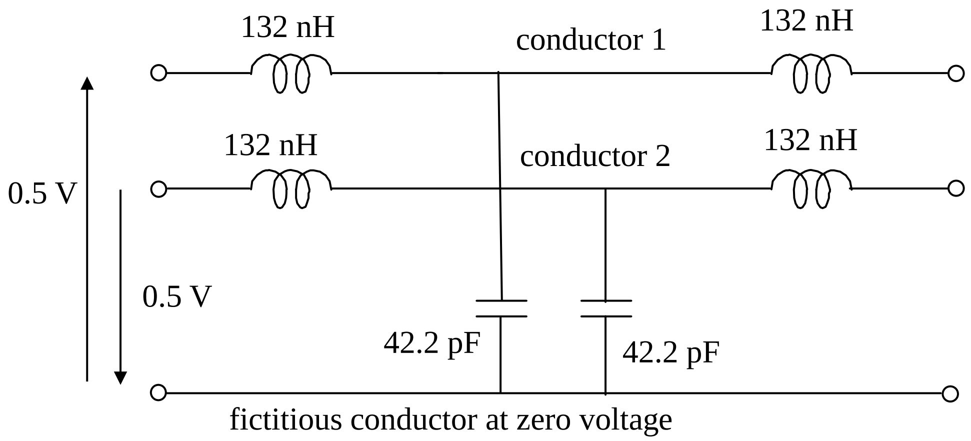
Figure 9 Assigning values to the inductors and capacitors of the circuit model

The worksheet also compares the results with that derived from a book on Electromagnetic Theory. Since the capacitance of two 42.24 pF capacitors in series is 21.12 pF, this comparison validates the technique described above.

Circuit model

The circuit model for the assembly-under-review is illustrated by Figure 10. The inductors in this model are assigned values of 132 nH, since each conductor is represented as a T-network. The fictitious conductor at zero volts has no physical existence, but is an essential part of the model.

It is possible to simplify the model further, by invoking the concept of equivalent circuits, but this action would bring with it the assumption that the return conductor can be represented as a zero-volt surface



Note: resistors are not shown since this model assumes the assembly is lossless

Figure10 Circuit model of 1 metre of twin conductor cable

Conclusion

The concept of the composite conductor has been introduced. The technique involves the use of elemental conductors to simulate the current distribution at the surface of each composite conductor they represent. It provides an analytical tool which can be used to assign accurate values to the inductance and capacitance of those conductors. The value derived for the capacitance between the conductors has been shown to be identical to the value derived in textbooks on electromagnetic theory.

The technique can be developed to derive values for the components of the circuit model which simulates any conductor assembly, given information on the geometry of that assembly. There is no need to invoke the complexities of full-field analysis.